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Explicit construction of automorphic forms on $Sp(1,q)$

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0 Introduction

In [Ar-1] and [Ar-2] Arakawa initiated the study of certain non-holomorphic automorphic forms on the real symplectic group $Sp(1,q)$ of signature $(1+,q-)$. He defined them as automorphic forms on $Sp(1,q)$ with the reproducing kernel function given by the matrix coefficient of some discrete series representation. That discrete is known as an example of quaternionic discrete series in the sense of Gross-Wallach [G-W]. Arakawa's definition deals only with bounded automorphic forms, assuming the integrability of the discrete series. In [N-1] we reformulated Arakawa's notion of the automorphic forms by using representation theoretic terminologies, without assuming the boundedness of the forms or the integrability of the discrete series. In other words we understood them as automorphic forms on $Sp(1,q)$ generating quaternionic discrete series.

In this note we provide three kinds of explicit constructions given in [N-2] for these automorphic forms. More precisely we construct Eisenstein series, Poincaré series and theta series for them. As for the construction by theta series, we consider the theta lifting from elliptic cusp forms to automorphic forms on $Sp(1,q)$ formulated by Arakawa in his unpublished note. This work was inspired by Kudla lifting, i.e. the theta lifting from elliptic modular forms to holomorphic automorphic forms on $SU(1,q)$ (cf. [Ku]). The fundamental tool for our results is the Fourier expansion of our automorphic forms developed in [N-1]. By virtue of it we can prove that our Eisenstein series and Poincaré series form a basis of the space of automorphic forms generating quaternionic discrete series and that the images of Arakawa lifting are bounded automorphic forms generating such discrete series for an arbitrary q. The latter result is a generalization of Arakawa's work on the lifting, which proves the case of $q=1$ in a different method.

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1 Basic notations and the definition of our automorphic forms

Throughout this note let $\mathbb{H}$ denote the Hamilton quaternion algebra with the standard basis $\{1, i, j, k\}$ and let tr (resp. ν) the reduced trace (resp. reduced norm) of $\mathbb{H}$. For $\xi \in \mathbb{H}$ we put $d(\xi) := \sqrt{\nu(\xi)}$.

Let G be the real symplectic group $Sp(1, q)$ of signature $(1+, q-)$ given by

$$Sp(1,q) := \{g \in M_{q+1}(\mathbb{H}) \mid {}^t\bar{g}Qg = Q\},$$

where

$$Q = \begin{cases} \begin{pmatrix} -S & & \\ & 0 & 1 \\ & 1 & 0 \end{pmatrix} & (q > 1) \\ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & (q = 1) \end{cases}$$

with a positive definite quaternion Hermitian matrix S of degree $q-1$. From now on, we fix a definite quaternion algebra B over $\mathbb{Q}$ contained in $\mathbb{H}$ and assume $S \in M_{q-1}(B)$.

This simple Lie group G acts on the quaternion hyperbolic space

$$H := \begin{cases} \{z = (w, \tau) \in \mathbb{H}^{q-1} \times \mathbb{H} \mid \operatorname{tr} \tau - wS^t\bar{w} > 0\} & (q > 1) \\ \{z \in \mathbb{H} \mid \operatorname{tr}(z) > 0\} & (q = 1) \end{cases}$$

via the linear fractional transformation

$$g \cdot z := \begin{cases} (a_1 w + b_1 \tau + c_1, a_2 w + b_2 \tau + c_2)\mu(g,z)^{-1} & (q > 1) \\ (a_1 z + b_1)\mu(g,z)^{-1} & (q = 1) \end{cases},$$

where

$$g = \begin{cases} \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} & (q > 1) \\ \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} & (q = 1) \end{cases}$$

with $a_1 \in M_{q-1}(\mathbb{H})$ $b_1, c_1, {}^t a_2, {}^t a_3 \in {}^t\mathbb{H}^{q-1}$ $b_2, b_3, c_2, c_3 \in \mathbb{H}$ (resp. $a_1, a_2, b_1, b_2 \in \mathbb{H}$) for $q > 1$ (resp. $q = 1$) and the automorphic factor

$$\mu(g,z) := \begin{cases} a_3 w + b_3 \tau + c_3 & (q > 1) \\ a_2 z + b_2 & (q = 1) \end{cases}.$$

We set $K := \{g \in G \mid g \cdot z_0 = z_0\}$ with $z_0 := \begin{cases} (0,1) & (q>1) \\ 1 & (q=1) \end{cases}$. This is isomorphic to $Sp^*(q) \times Sp^*(1)$, where $Sp^*(q)$ denotes the compact real form of the complex symplectic group of degree q. This forms a maximal compact subgroup of G.

Hereafter κ denotes a positive integer. For such κ we define a representation (τ_κ, V_κ) of K by

$$\tau_\kappa(k) := \sigma_\kappa(\mu(k, z_0)) \quad (k \in K),$$

where $(\sigma_\kappa, V_\kappa)$ is the κ-th symmetric tensor representation of $Sp^*(1)$. For this representation we note $\tau_\kappa \simeq \mathrm{id}_{Sp(q)} \boxtimes \sigma_\kappa$. In what follows we fix an K-invariant inner product $(*,*)_\kappa$ of V_κ with respect to τ_κ, and denote by $\| * \|_\kappa$ the norm of V_κ induced by $(*,*)_\kappa$.

For $\kappa > 2q-1$ let π_κ be the discrete series representation of G with minimal K-type τ_κ. This π_κ is known as an example of "quaternionic discrete series" introduced by B.Gross and N.Wallach [G-W]. When $\kappa > 4q$, π_κ is integrable.

In the subsequent argument we need $\omega_\kappa : G \to End(V_\kappa)$ defined by

$$\omega_\kappa(g) := \sigma_\kappa(a(g))^{-1} \nu(a(g))^{-1},$$

where

$$a(g) := \frac{1}{2}(\tau(g \cdot z_0) + 1)\mu(g, z_0)$$

with

$$\tau(z) := \begin{cases} \text{the second entry of } z & (q > 1) \\ z & (q = 1) \end{cases}$$

for $z \in H$. This ω_κ is the matrix coefficient of π_κ.

Now we state the definition of the automorphic forms in our concern:

Definition 1.1. Let $\kappa > 2q-1$. For an arithmetic subgroup Γ of G, $\mathcal{A}(\Gamma\backslash G, \pi_\kappa)$ denotes the space of all V_κ-valued C^∞-functions f on G satisfying:

(1) $f(\gamma g k) = \tau_\kappa(k)^{-1} f(g)\ \forall(\gamma, g, k) \in \Gamma \times G \times K$,

(2) $\langle \text{coeff. of } f(*g) \mid g \in G \rangle \simeq \pi_\kappa$ as $(\mathfrak{g}, K)$-modules ($\mathfrak{g}$:Lie algebra of G),

(3) f is of moderate growth when $q = 1$.

Furthermore we put $\mathcal{A}_0(\Gamma\backslash G, \pi_\kappa) := \{f \in \mathcal{A}(\Gamma\backslash G, \pi_\kappa) \mid f\text{:bounded}\}$.

Remark 1.2. (1) When $q > 1$, f automatically satisfies the moderate growth condition. We call this property *Koecher principle* (cf. [N-1, Theorem 7.1])

(2) The second condition can be replaced by

$$D_\kappa \cdot f = 0 \quad (D_\kappa\text{:Schmid operator})$$

(cf. [N-1, Theorem 8.2]). For the definition of the Schmid operator see [Kn, Chap.XII, §10, Problems] and [N-1, Definition 5.2].
(3) Moreover, assuming that f is bounded, we can replace this condition by

$$c_\kappa \int_G \omega_\kappa(g^{-1}h) f(g) dg = f(h)$$

(cf. [N-1, Theorem 8.7]), where $c_\kappa = \frac{d_\kappa}{\kappa+1}$ with the formal degree d_κ of π_κ. Under the assumption we can verify that f is cuspidal (cf. [Ar-2, Proposition 3.1]).

2 Fourier expansion

In this section we write down the Fourier expansion for $\mathcal{A}(\Gamma\backslash G, \pi_\kappa)$, developed in [Ar-2] and [N-1]. It plays a crucial role to obtain our results.

We introduce notations necessary to describe the expansion. Let

$$N := \begin{cases} \left\{ n(w,x) := \begin{pmatrix} 1_{q-1} & 0 & w \\ {}^t\bar{w}S & 1 & \frac{1}{2}{}^t\bar{w}Sw + x \\ 0 & 0 & 1 \end{pmatrix} \middle| \begin{array}{c} w \in {}^t\mathbb{H}^{q-1} \\ x \in X \end{array} \right\} & (q > 1) \\ \left\{ n(x) := \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \middle| \ x \in X \right\} & (q = 1) \end{cases}$$

with

$$X := \{x \in \mathbb{H} \mid \operatorname{tr} x = 0\}$$

and let

$$A := \begin{cases} \left\{ a = a_y := \begin{pmatrix} 1_{q-1} & & \\ & \sqrt{y} & \\ & & \sqrt{y}^{-1} \end{pmatrix} \middle| \ y \in \mathbb{R}_+ \right\} & (q > 1) \\ \left\{ a = a_y := \begin{pmatrix} \sqrt{y} & \\ & \sqrt{y}^{-1} \end{pmatrix} \middle| \ y \in \mathbb{R}_+ \right\} & (q = 1) \end{cases}.$$

Then G admits the Iwasawa decomposition $G = NAK$.

We fix $\mathbb{Q}$-structure $G(\mathbb{Q}) := G \cap M_{q+1}(B)$ and let $\Gamma \subset G(\mathbb{Q})$ be an arithmetic subgroup of G. For the standard proper parabolic subgroup P of G we put $P(\mathbb{Q}) := P \cap G(\mathbb{Q})$. We denote by Ξ a complete set of representatives of $\Gamma\backslash G(\mathbb{Q})/P(\mathbb{Q})$, i.e. the set of Γ-cusps. For $c \in \Xi$ we set

$$\begin{aligned} N_{\Gamma,c} &:= c^{-1}\Gamma c \cap N, \\ X_{\Gamma,c} &:= \{x \in X \mid n(0,x) \in N_{\Gamma,c}\}, \\ \Lambda_c &:= \{\lambda \in {}^t\mathbb{H}^{q-1} \mid n(\lambda, x_\lambda) \in N_{\Gamma,c}, \ \exists x_\lambda \in X\}, \end{aligned}$$

where the lattice Λ_c is defined only when $q > 1$.

When $q > 1$ we introduce a space of theta functions for $\xi \in X^*_{\Gamma,c} \setminus \{0\}$ defined by

$$\Theta_{\xi,c} := \left\{ \theta \in C({}^t\mathbb{H}^{q-1}) \;\middle|\; \begin{array}{l} \theta(w+\lambda) = \mathbf{e}(\operatorname{tr}\xi({}^t\bar{w}S\lambda - x_\lambda))\theta(w) \ \forall \lambda \in \Lambda_c \\ \int_{{}^t\mathbb{H}^{q-1}} k_\xi(w',w)\theta(w')dw' = \theta(w) \end{array} \right\},$$

where

$$k_\xi(w',w) := \Delta(S)2^{4(q-1)}\nu(\xi)^{q-1}\exp(-2\pi d(\xi){}^t\overline{(w-w')}S(w-w'))\mathbf{e}(-\operatorname{tr}(\xi{}^t\bar{w}'Sw))$$

with $d(Sw) = \Delta(S)^2 dw$. This space has an inner product given by

$$(\theta_1, \theta_2)_{\xi,c} := \int_{\mathbb{H}^{q-1}/L_c} \theta_1(w)\overline{\theta_2(w)}dw.$$

For each $\xi \in X^*_{\Gamma,c} \setminus \{0\}$ we fix $u_\xi \in \{a \in \mathbb{H} \mid \nu(a) = 1\}$ such that $u_\xi i \bar{u}_\xi = \xi/d(\xi)$. Then we have

Proposition 2.1. *The Fourier expansion of $f \in \mathcal{A}(\Gamma\backslash G, \pi_\kappa)$ at a cusp $c \in \Xi$ is written as follows:*

$$f(cn(w,x)a) = \sum_{i=0}^{\kappa} C_i^f y^{\frac{\kappa}{2}+1} v_{\kappa,i} + y^{\frac{\kappa}{2}+1} \sum_{\xi \in X^*_{\Gamma,c}\setminus\{0\}} a_\xi^f(w) e^{-4\pi d(\xi) y} \mathbf{e}(\operatorname{tr}\xi x)\sigma_\kappa(u_\xi)v_{\kappa,\kappa} \quad (q > 1),$$

$$f(cn(x)a) = \sum_{i=0}^{\kappa} C_i^f y^{\frac{\kappa}{2}+1} v_{\kappa,i} + y^{\frac{\kappa}{2}+1} \sum_{\xi \in X^*_{\Gamma,c}\setminus\{0\}} C_\xi^f e^{-4\pi d(\xi) y} \mathbf{e}(\operatorname{tr}\xi x)\sigma_\kappa(u_\xi)v_{\kappa,\kappa} \quad (q = 1),$$

where

- *$\{v_{\kappa,i}\}_{0\le i\le\kappa}$ is a fixed basis of V_κ with a highest weight vector $v_{\kappa,\kappa}$ satisfying some standard relation (for its precise meaning see* [N-1, (2.1) (2.2) (2.3)]),
- $a_\xi^f(w) \in \Theta_{\xi,c}$, and C_ξ^f (resp. C_i^f) is a constant dependent only on (ξ, f) (resp. (i, f)).

For this proposition see [Ar-2, Theorem 6.1] and [N-1, Theorem 6.3, §9].

3 Eisenstein-Poincaré series

This section provides explicit constructions of the automorphic forms in $\mathcal{A}(\Gamma\backslash G, \pi_\kappa)$ by Eisenstein series and Poincaré series.

We first consider the Eisenstein series. For $s \in \mathbb{C}$ and $v \in V_\kappa$ we set

$$W_{s,v}(n(w,x)a_yk) := \tau_\kappa(k)^{-1}y^s v,$$

where we replace $n(w,x)$ by $n(x)$ when $q = 1$. When $s = \frac{\kappa}{2}+1$ for $\kappa > 2q-1$ this is a generalized Whittaker functions for π_κ with K-type (τ_κ, V_κ) attached to the trivial representation of N (cf. [N-1, Theorem 5.5]). For a representative c of Ξ we define an Eisenstein series at a cusp c as follows:

$$E_{c,v}(g;s) := \sum_{g\in\Gamma\cap cNc^{-1}\backslash\Gamma} W_{s,v}(c^{-1}\gamma g).$$

Theorem 3.1. *$E_{c,v}(g;s)$ converges absolutely and uniformly on any compact subset of G if* $\mathrm{Re}(s) > 2q+1$. *In particular, $E_{c,v}(g;\frac{\kappa}{2}+1) \in \mathcal{A}(\Gamma\backslash G, \pi_\kappa)$ when $\kappa > 4q$.*

Proof. The convergence range is due to the Godement's criterion on the convergence of Eisenstein series (cf. [B-1, Lemma 11.1 and Theorem 12.1]), which was pointed out by Arakawa in [Ar-2, §6.2]. For the rest of the assertion we recall that $D_\kappa \cdot W_{\frac{\kappa}{2}+1,v}(g) = 0$ (cf. [Y, Proposition 2.1, Theorem 2.4]) with the Schmid operator D_κ (for D_κ see Remark 1.2 (2)) and that $E_{c,v}(g,s)$ defines a smooth automorphic form on G (cf. [Ha, Chap.II, §2]). These imply $D_\kappa \cdot E_{c,v}(g;\frac{\kappa}{2}+1) = 0$. In view of Remark 1.2 (2) we see $E_{c,v}(g;\frac{\kappa}{2}+1) \in \mathcal{A}(\Gamma\backslash G, \pi_\kappa)$. □

Next we consider the construction by Poincaré series. For $\xi \in X^*_{\Gamma,c} \setminus \{0\}$, $\theta \in \Theta_{\xi,c}$ and $v \in V_\kappa$, we put

$$W_{\theta,v}(n(w,x)a_yk) := \tau_\kappa(k)^{-1}\theta(w)y^{\frac{\kappa}{2}+1}e^{-4\pi d(\xi)y}\mathbf{e}(\mathrm{tr}\,\xi x)U_\xi(v) \quad (q>1),$$
$$W_{\xi,v}(n(x)a_yk) := \tau_\kappa(k)^{-1}y^{\frac{\kappa}{2}+1}e^{-4\pi d(\xi)y}\mathbf{e}(\mathrm{tr}\,\xi x)U_\xi(v) \quad (q=1),$$

where $U_\xi \in \mathrm{End}(V_\kappa)$ is the projection from V_κ to $\mathbb{C}\sigma_\kappa(u_\xi)v_{\kappa,\kappa}$.

Using this, we define a Poincaré series at a representative c of Ξ by

$$P_{c,v}(g;\theta) := \sum_{\Gamma\cap cNc^{-1}\backslash\Gamma} W_{\theta,v}(c^{-1}\gamma g) \quad (q>1),$$
$$P_{c,v}(g;\xi) := \sum_{\Gamma\cap cNc^{-1}\backslash\Gamma} W_{\xi,v}(c^{-1}\gamma g) \quad (q=1).$$

Theorem 3.2. *Suppose $\kappa > 4q$.*
(1) *For each fixed $c \in \Xi$, $\{P_{c,v}(g,\theta) \mid v \in V_\kappa,\ \theta \in \Theta_{\xi,c},\ \xi \in X^*_{\Gamma,c} \setminus \{0\}\}$ (resp. $\{P_{c,v}(g;\xi) \mid v \in V_\kappa,\ \xi \in X^*_{\Gamma,c} \setminus \{0\}\}$) spans $\mathcal{A}_0(\Gamma\backslash G, \pi_\kappa)$ when $q > 1$ (resp. $q = 1$).*
(2) *Assume that $W_{\frac{\kappa}{2}+1,v}$ is left $c^{-1}\Gamma c \cap P$-invariant for each $c \in \Xi$. Then $\{P_{c,v}(g;\theta), E_{c,v}(g;\frac{\kappa}{2}+1) \mid c \in \Xi,\ v \in V_\kappa,\ \theta \in \Theta_{\xi,c},\ \xi \in X^*_{\Gamma,c} \setminus \{0\}\}$ (resp. $\{P_{c,v}(g;\xi), E_{c,v}(g;\frac{\kappa}{2}+1) \mid c \in \Xi,\ v \in V_\kappa,\ \xi \in X^*_{\Gamma,c} \setminus \{0\}\}$) spans $\mathcal{A}(\Gamma\backslash G, \pi_\kappa)$ when $q > 1$ (resp. $q = 1$).*

Proof. We omit the case of $q = 1$ since the proof is similar to the case of $q > 1$. Let us consider the first assertion. We can verify the boundedness of $P_{c,v}(g,\theta)$ by following similarly the standard argument on the absolute convergence of Poincaré series by A.Borel [B-2, Theorem 6.1]. In fact, such argument deduces

$$\|P_{c,v}(g;\theta)\|_\kappa \le M \cdot \int_{\Gamma \cap cNc^{-1}\backslash G} \|W_{\theta,v}(c^{-1}h)\|_\kappa dh < \infty,$$

where M denotes a constant not dependent on g. Moreover we can show that $W_{\theta,v}$ satisfies the property in Remark 1.2 (3) (cf. [N-1, Lemma 8.6]). Thus we see $P_{c,v}(g,\theta) \in \mathcal{A}_0(\Gamma\backslash G, \pi_\kappa)$ in view of Remark 1.2 (3). The proof for the exhaution of $\mathcal{A}_0(\Gamma\backslash G, \pi_\kappa)$ by the Poincaré series is also standard.

As for the second assertion, it is important to note that the constant term of $E_{c,v}$ at $c' \in \Xi$ is equal to

$$\begin{cases} [c^{-1}\Gamma c \cap P : N_{\Gamma,c}]y^{\frac{\kappa}{2}+1}v & (c' = c), \\ 0 & (c' \neq c) \end{cases}$$

under the assumption on $W_{\frac{\kappa}{2}+1,v}$. Then we see that the assertion is an immediate consequence of the Fourier expansion in Proposition 2.1. $\square$

4 Arakawa's theta lifting

In this section we consider the theta lifting from elliptic cusp forms to automorphic forms on $Sp(1,q)$ formulated by Arakawa. For this lifting we should note that $(SL_2(\mathbb{R}), Sp(1,q))$ does not form any reductive dual pair unless $q = 1$. Thus, when $q > 1$, we can not provide the usual formulation of the theta lifting for a pair $(SL_2(\mathbb{R}), Sp(1,q))$ by means of the Weil representation. In order to overcome this difficulty Arakawa regarded $Sp(1,q)$ as a subgroup of $SO(4,4q)$ and considered the restriction of a theta series on a dual pair $SL_2(\mathbb{R}) \times SO(4,4q)$ to a non-dual pair $SL_2(\mathbb{R}) \times Sp(1,q)$ for the formulation of the lifting. More precisely, he constructed a theta series on $\mathfrak{h} \times SO(4,4q)$ after Shintani [Shin] and restrict it to $\mathfrak{h} \times Sp(1,q)$ to formulate the lifting, where $\mathfrak{h}$ denotes the complex upper half plane.

Let us introduce the theta series on $\mathfrak{h} \times G$ to construct the lifting. We provide two quaternion Hermitian forms

$$(*,*)_Q : \mathbb{H}^{q+1} \times \mathbb{H}^{q+1} \ni (x,y) \mapsto (x,y)_Q := \mathrm{tr}(xQ{}^t\bar{y}) \in \mathbb{R}$$
$$(*,*)_R : \mathbb{H}^{q+1} \times \mathbb{H}^{q+1} \ni (x,y) \mapsto (x,y)_R := \mathrm{tr}(xR{}^t\bar{y}) \in \mathbb{R}$$

with

$$R := \begin{cases} \begin{pmatrix} S & \\ & 1_2 \end{pmatrix} & (q > 1) \\ 1_2 & (q = 1) \end{cases} \quad \text{(majorant of } Q).$$

We note that $(*, *)_Q$ is regarded as a symmetric bilinear form on $\mathbb{R}^{4(q+1)} \simeq \mathbb{H}^{q+1}$. Via this form we can consider $Sp(1, q)$ as a subgroup of the special orthogonal group $SO(4, 4q)$ of signature $(4+, 4q-)$.

For $z = s + \sqrt{-1}t \in \mathfrak{h}$ and $x = (*, x_q, x_{q+1}) \in \mathbb{H}^{q+1}$ we put

$$F_z(x) := \sigma_\kappa(x_q + x_{q+1})\mathbf{e}\left(\frac{1}{2}(s(x, x)_Q + \sqrt{-1}t(x, x)_R)\right).$$

Then we give a theta series on $\mathfrak{h} \times Sp(1, q)$ defined by Arakawa as follows:

$$\theta(z, g) := t^{2q} \sum_{l \in L} F_z(l^t \bar{g}^{-1}),$$

where $L := \mathcal{O}^{q+1}$ with a fixed maximal order $\mathcal{O}$ of B.

From now on, we assume

$$S = \mathrm{diag}(\alpha_1, \alpha_2, \ldots, \alpha_{q-1}) \quad (\alpha_i \in \mathbb{Z}_{>0})$$

and let $N \in \mathbb{Z}_{>0}$ be divisible by the least common multiplier of $\{2, d(B), \alpha_1, \ldots, \alpha_{q-1}\}$, where $d(B)$ denotes the product of ramified primes of B. For such N we set

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \;\middle|\; c \equiv 0 \mod N \right\}.$$

Furthermore we fix an arithmetic subgroup

$$\Gamma := \{\gamma \in G(\mathbb{Q}) \mid L^t\bar{\gamma} = L\}.$$

Then we have

Proposition 4.1 (Arakawa). *Let the notations be as above. Then $\theta(z, g)$ satisfies*

$$\theta(\delta(z), \gamma g k) = J(\delta, z)^{\kappa+2-2q}\theta(z, g)\tau_\kappa(k)$$

for $(\delta, \gamma, k) \in \Gamma_0(N) \times \Gamma \times K$.

The most difficult step of this proposition is the transformation formula with respect to $\Gamma_0(N)$. It is settled by using basically Shintani's argument on the transformation formula of a theta series attached to the Weil representation (cf. [Shin, Proposition 1.6]).

Now we formulate Arakawa's theta lifting by using the theta series $\theta(z,g)$. Let $S_{\kappa-2q+2}(\Gamma_0(N))$ be the space of elliptic cusp forms of weight $\kappa-2q+2$ with respect to $\Gamma_0(N)$. For $f \in S_{\kappa-2q+2}(\Gamma_0(N))$ we put

$$\Phi(g,f) := \int_{\Gamma_0(N)\backslash\mathfrak{h}} \theta(z,g)^* f(z) t^{\kappa-2q} ds dt,$$

which is $\mathrm{End}(V_\kappa)$-valued.

Proposition 4.2 (Arakawa). *The* $\mathrm{End}(V_\kappa)$*-valued function* $\Phi(g,f)$ *converges absolutely and uniformly on any compact subset of* G*. Moreover* $\Phi(g,f)$ *is of moderate growth.*

Proof. We can prove this by the reasoning similar to [O, §5, 2]. The point is to estimate the norm of $\theta(z,g)$ by Epstein zeta function attached to R. □

Now we state our theorem on the theta lifting.

Theorem 4.3 (Arakawa, N). *Let* $\kappa > 4q+2$*. For* $v \in V_\kappa$,

$$\Phi(g,f)\cdot v \in \mathcal{A}_0(\Gamma\backslash G, \pi_\kappa).$$

The rest of this note is devoted to overviewing the proofs of this theorem by Arakawa and us. The first step of it is to consider the lifting of elliptic Poincaré series

$$G_m(z) := \sum_{\delta\in\Gamma_\infty\backslash\Gamma_0(N)} J(\gamma,z)^{-(\kappa-2q+2)}\mathbf{e}(m(\delta(z))) \quad (z\in\mathfrak{h})$$

for each positive integer m, where $\Gamma_\infty := \left\{\pm\begin{pmatrix}1 & h\\ 0 & 1\end{pmatrix} \;\middle|\; h\in\mathbb{Z}\right\}$ and $J(\gamma,z)$ means the standard $\mathbb{C}$-valued automorphic factor. It suffices to prove $\Phi(g,G_m)\cdot v \in \mathcal{A}_0(\Gamma\backslash G,\pi_\kappa)$ for every m since $\{G_m \mid m\in\mathbb{Z}_{>0}\}$ spans $S_{\kappa-2q+2}(\Gamma_0(N))$. Arakawa obtained

$$\Phi(g,G_m) = (2m)^{-(1+\frac{\kappa}{2})}\frac{\Gamma(\kappa+1)}{(2\pi)^{\kappa+1}}\Omega_m(g)$$

with

$$\Omega_m(g) := \sum_{\substack{l\in L\\ (l,l)_Q=2m}} \omega_\kappa(p_l^{-1}g)\sigma_\kappa\left(\frac{l_{q+1}}{d(l_{q+1})}\right)^{-1},$$

where, for $l=(\tilde{l},l_q,l_{q+1})\in\mathbb{H}^{q-1}\times\mathbb{H}\times\mathbb{H}$ ($l=(l_q,l_{q+1})\in\mathbb{H}\times\mathbb{H}$ when $q=1$) with the positive $(l,l)_Q$, we can take a unique $p_l\in NA$ such that

$$p_l\cdot z_o = \begin{cases}({}^t(\overline{l_{q+1}^{-1}\tilde{l}}), \overline{l_{q+1}^{-1}l_q}) \in H & (q>1)\\ \overline{l_{q+1}^{-1}l_q}\in H & (q=1)\end{cases}.$$

This converges absolutely and uniformly on any compact subset of G when $\kappa > 4q+2$.

Then the next step is to prove that $\Omega_m(g)\cdot v$ belongs to $\mathcal{A}_0(\Gamma\backslash G, \pi_\kappa)$. For this step Arakawa's idea and ours are different. We first explain Arakawa's proof, which deals with the case of $q=1$.

Let $q=1$ and put $L(m) := \{l \in L \mid (l,l)_Q = 2m\}$ for every positive integer m. Arakawa showed

$$\sharp(L(m)/\Gamma) < \infty,$$

where Γ acts on $L(m)$ via

$$\Gamma \ni \gamma : L(m) \ni l \mapsto l^t\overline{\gamma}^{-1} \in L(m).$$

This is verified by considering the embedding

$$L(m)/\Gamma \ni (l \mod \Gamma) \mapsto (p_l \cdot z_0 \mod \Gamma) \in \Gamma\backslash H$$

together with an explicit description of a fundamental domain of $\Gamma\backslash H$. As a result, it can be proved that

$$\Omega_m(g) = \text{const.} \times \sum_{l\in L(m)/\Gamma} K_\kappa^\Gamma(p_l, g),$$

where $K_\kappa^\Gamma(g_1,g_2) := \sum_{\gamma\in\Gamma}\omega_\kappa(g_1^{-1}\gamma g_2)$ is the Godement kernel function for $\mathcal{A}_0(\Gamma\backslash G,\pi_\kappa)$. Since $K_\kappa^\Gamma(g_0,g)\cdot v \in \mathcal{A}_0(\Gamma\backslash G,\pi_\kappa)$ for $v \in V_\kappa$ and a fixed $g_0 \in G$, this formula of $\Omega_m(g)$ implies the theorem for the case of $q=1$.

Next we explain our idea. Our method is to use Fourier expansion in Proposition 2.1. To obtain the Fourier expansion of $\Omega_m(g)\cdot v$ we need

Lemma 4.4 (Fourier Transformation of ω_κ). *Let $\xi \in X_\mathbb{R}$. When $q>1$*

$$\int_X \omega_\kappa(p_v^{-1}n(w,x)a)\mathbf{e}(-(\operatorname{tr}\xi(x)))dx = d(\xi)^{\kappa-1}\delta_\xi(v,m,\kappa)k_\xi^0(w_v,w)y^{\frac{\kappa}{2}+1}\exp(-4\pi d(\xi)y)\cdot U(\xi)$$

and when $q=1$

$$\int_X \omega_\kappa(p_v^{-1}n(x)a)\mathbf{e}(-(\operatorname{tr}\xi(x)))dx = d(\xi)^{\kappa-1}\delta_\xi(v,m,\kappa)y^{\frac{\kappa}{2}+1}\exp(-4\pi d(\xi)y)\cdot U(\xi),$$

where

- $(w_l,\tau_l) := p_l(z_0) \in H$,
- $k_\xi^0(w',w) := (\Delta(S)2^{4(q-1)}\nu(\xi)^{q-1})^{-1}k_\xi(w',w)$,
- *$U(\xi) \in End(V_\kappa)$ is the projection from V_κ onto $\mathbb{C}\cdot\sigma_\kappa(u_\xi)v_{\kappa,\kappa}$,*

•

$$\delta_\xi(l,m,\kappa) := \frac{2\pi^2(4\pi)^{\kappa-1}}{\kappa!}\left(\frac{2m}{\nu(l_{q+1})}\right)^{\frac{\kappa}{2}+1}\exp\left(-\frac{2\pi d(\xi)m}{\nu(l_{q+1})}\right)\mathbf{e}(-\frac{1}{2}\operatorname{tr}\xi(\tau_l-\bar{\tau}_l)).$$

In paticular, this is equal to 0 *when* $\xi = 0$.

This is a consequence of Arakawa's Fourier transformation formula of ω_κ (cf. [Ar-1, Lemma 1.2]). In addition, we use

Lemma 4.5. *Let* $\kappa > 4q$. *If cuspidal automorphic forms on* G *of weight* τ_κ *with respect to* Γ *have the Fourier expansion of the form in Proposition* 2.1, *they belong to* $\mathcal{A}_0(\Gamma\backslash G,\pi_\kappa)$.

For this lemma see [N-2, Proposition 2.4].

By virtue of Lemma 4.4 we obtain a Fourier expansion of Ω_m at a cusp $c \in \Xi$:

$$\Omega_m(cn(w,x)a) = \frac{1}{\mathrm{vol}(X/X_{\Gamma,c})}\sum_{\xi\in X^*_{\Gamma,c}\backslash\{0\}}\theta_\xi(w)y^{1+\frac{\kappa}{2}}\exp(-4\pi d(\xi)y)\mathbf{e}(\operatorname{tr}(\xi x))\ (q>1),$$

$$\Omega_m(cn(x)a) = \frac{1}{\mathrm{vol}(X/X_{\Gamma,c})}\sum_{\xi\in X^*_{\Gamma,c}\backslash\{0\}}C^m_\xi y^{1+\frac{\kappa}{2}}\exp(-4\pi d(\xi)y)\mathbf{e}(\operatorname{tr}(\xi x))\ (q=1),$$

where

$$\mathrm{vol}(X_{\mathbb{R}}/X_{\Gamma,c}) := \text{the volume of the quotient } X_{\mathbb{R}}/X_{\Gamma,c}$$

$$\theta_\xi(w) := \sum_{\substack{v\in L_c/N_{\Gamma,c}\cap Z(N)\\ (v,v)_Q=2m}} d(\xi)^{\kappa-1}\delta_\xi(v,m,\kappa)k^0_\xi(w_v,w)\cdot U(\xi)\cdot\sigma_\kappa\left(\frac{v_{q+1}}{d(v_{q+1})}\right)^{-1}$$

$$C^m_\xi := \sum_{\substack{v\in L_c/N_{\Gamma,c}\cap Z(N)\\ (v,v)_Q=2m}} d(\xi)^{\kappa-1}\delta_\xi(v,m,\kappa)\cdot U(\xi)\cdot\sigma_\kappa\left(\frac{v_{q+1}}{d(v_{q+1})}\right)^{-1}$$

with the center $Z(N)$ of N and $L_c := L^t\bar{c}^{-1}$. Here the quotient L_c by $N_{\Gamma,c}\cap Z(N)$ is induced by the action of $c^{-1}\Gamma c$ on L_c, which is similar to the action of Γ on $L(m)$.

We can prove that $\theta_\xi(w)$ is a bounded function on $\mathbb{H}^{q-1}$ and that $\theta_\xi(w)$ satisfies the two conditions of $\Theta_{\xi,c}$ (cf. [N-2, Proposition 4.7 (1)]). Therefore the coefficients of $\theta_\xi(w)$ belong to $\Theta_{\xi,c}$. Then we see that $\Omega_m(g)\cdot v$ has the same Fourier expansion as in Proposition 2.1. Thus Lemma 4.5 implies $\Omega_m(g)\cdot v \in \mathcal{A}_0(\Gamma\backslash G,\pi_\kappa)$.

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